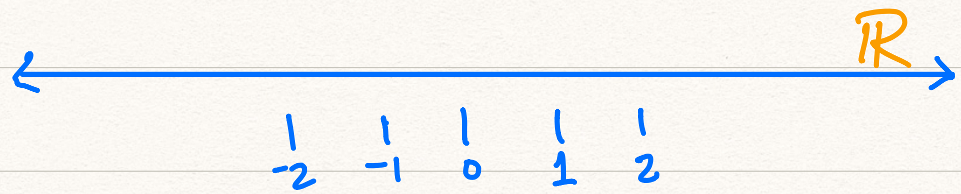
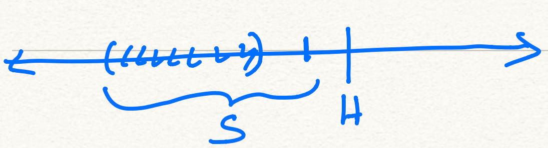
Intuitively, the completeness of $\mathbb{R}$ can be understood as follows:

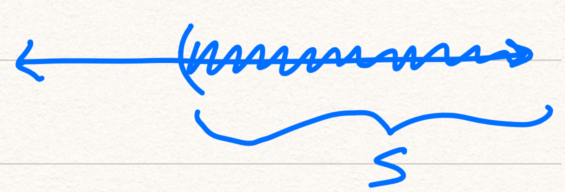
If $\mathbb{R}$ is represented as a straight line that extends indefinitely in both directions, the completeness of $\mathbb{R}$ corresponds to the continuous nature of the line (solid line, no gaps).



Bounds

- A set $S \subseteq \mathbb{R}$ is bounded above if there exists some $H \in \mathbb{R}$ such that $x \leq H$ for every $x \in S$. If H exists, it is called an upper bound of S.

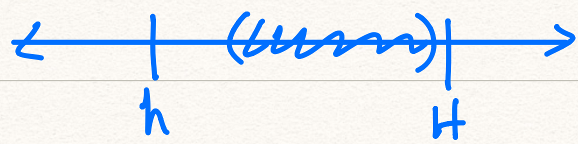




- A set $S \subseteq \mathbb{R}$ is bounded below if there exists some $h \in \mathbb{R}$ such that $h \leq x$ for every $x \in S$. If h exists, it is called a lower bound of S.

No lower bounds, no upper b.

- A set which is both bounded below and above is just said to be bounded.



Examples:

(i) $\{1, 2, 3\}$ Bounded

Bounded above : upper bounds 3, 10, 1000

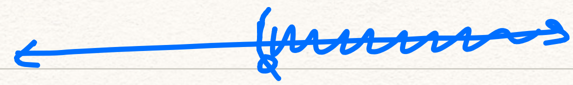
bounded below : lower bounds 1, -10, -1000

(ii) $\{x : 1 \leq x < 2\}$ $1.\overline{9} = 2$

Bounded above : $2, 3, 50,$ 1.9 not upper boud

1.91

Bounded below: $1, 0, -12, -1M$.

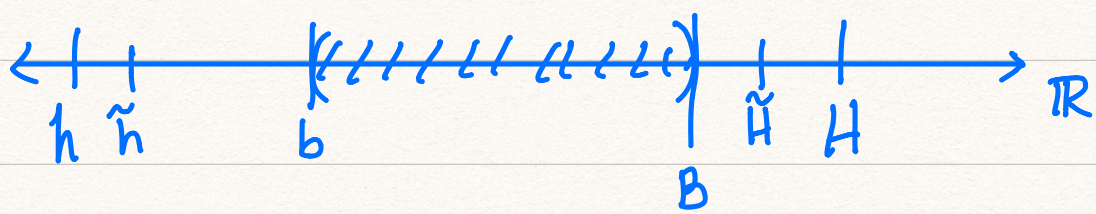
(iii) $\{x : x > 0\}$ 

Not bounded above because for eve H in the set, $H+1 > H$ and $H+1$ is also in the set

Bounded below: $0, -1, -15, \ldots$

Continuum property

- Every non-empty set of real numbers which is bounded above has a smallest upper bound. (Supremum $B = \sup S$)
- Every non-empty set of real numbers which is bounded below has a greatest lower bound. (infimum $b = \inf S$)



Examples:

(i) $\{1, 2, 3\}$

Supremum: 3

Infimum: 1

(ii) $\{x : 1 \leq x < 2\}$

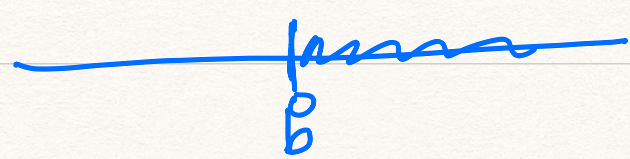
Infimum: 1

Supremum: 2

(iii) $\{x : x > 0\}$

Unbounded above $\Rightarrow$ no supremum

Infimum: 0



Maximum and minimum

If a non-empty set S is bounded above and $B = \sup S \in S$, then B is called the maximum of S.

If a non-empty set S is bounded below and $b = \inf S \in S$, then b is called the minimum of S.

Examples:

(i) $\{1, 2, 3\}$ Not an interval

Maximun? yes 3

Minimum? yes 1

(ii) $\{x : 1 \leq x < 2\}$ Interval

Maximun? No

Minimum? yes 1

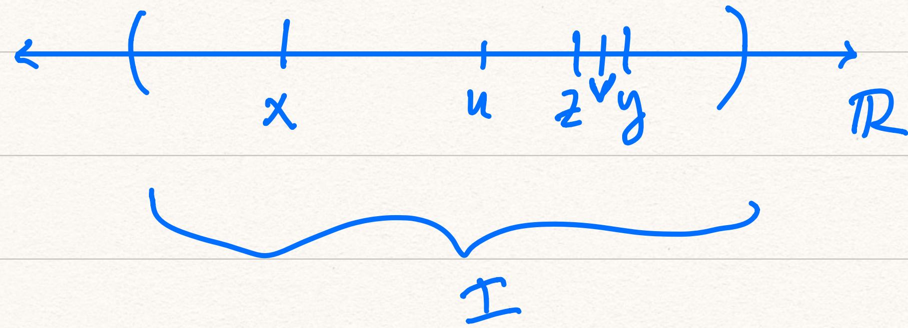
(iii) $\{x : x > 0\}$ Interval

Max? No, not bounded above.

Min? No

Intervals

An interval I is a set of real numbers with the property that, if $x \in I$ and $y \in I$ and $x \le z \le y$, then $z \in I$.



Notation:

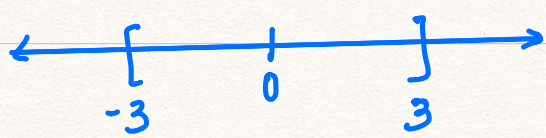
- Bounded intervals:

1. $(a,b) = \{x : a < x < b\}$
2. $[a,b] = \{x : a \le x \le b\}$
3. $[a,b) = \{x : a \le x < b\}$
4. $(a,b] = \{x : a < x \le b\}$

- **Unbounded intervals:**

5. $(a, \infty) = \{x : x > a\}$
6. $[a, \infty) = \{x : x \geq a\}$
7. $(-\infty, b) = \{x : x < b\}$
8. $(-\infty, b] = \{x : x \leq b\}$.

<u>Example</u>: Identify the set $A = \{x : |x| \leq 3\}$.



$A = [-3, 3]$

Case I: if $x > 0$ then $|x| = x$

so $|x| \leq 3 \Rightarrow x \leq 3$

Case II: if $x \leq 0$, then $|x| = -x$

so $|x| \leq 3 \Rightarrow -x \leq 3$ or $-3 \leq x$

Putting case I and case II together we get

$-3 \leq x \leq 3$

Example: Identify the set

$$B = \{x : (x-1)(x-2)(x-3) < 0\} = (-\infty, 1) \cup (2, 3)$$

	$(-\infty, 1)$	$(1, 2)$	$(2, 3)$	$(3, \infty)$
$x-1$	$-$	$+$	$+$	$+$
$x-2$	$-$	$-$	$+$	$+$
$x-3$	$-$	$-$	$-$	$+$
$(x-1)(x-2)(x-3)$	$-$	$+$	$-$	$+$